Random spread on the family of small-world networks

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We present analytical and numerical results of a random walk on the family of small-world graphs. The average access time shows a crossover from regular to random behavior with increasing distance from the starting point of the random walk. We introduce an independent step approximation, which enables us to obtain analytic results for the average access time. We observe a scaling relation for the average access time in the degree of the nodes. The behavior of the average access time as a function of p shows striking similarity with that of the characteristic length of the graph. This observation may have important applications in routing and switching in networks with a large number of nodes.

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I. INTRODUCTION

The small-world network exhibits unusual connection properties. On one hand it shows strong clustering, like regular graphs, and on the other hand it shows a very small average shortest path between any two nodes, like random graphs. Watts and Strogatz have proposed a simple model to describe small-world networks [1]. The model gives a prescription for generating a one-parameter family of graphs, ranging from highly clustered (regular) graphs to random graphs.

Various properties of this model have been studied [2–14]. The spread and percolation properties investigated in Refs. [2–5] deal with the spread of information (disease) along the shortest path in the graph or the spread along the spanning tree. In Ref. [14] the diffusion process and the dispersion relations on small-world networks are studied.

In this paper, we study random walks on the family of small-world networks. Such a random walk corresponds to random spread of information on the network. In any realistic application of the spread on a graph, we expect the spread to be somewhere in between the two extremes, viz., the shortest path and the random walk. For example, in Milgram's experiment [15], which studies the connection properties of social networks, the path of a letter from a randomly chosen point to a fixed target is traced. The only condition imposed on the transfer of letter is that the letter should be given to a person whom the sender knows by first name. The path followed by such a letter would have both random and shortest path elements in it. Another example is the path of an internet protocol packet which follows a similar algorithm for forwarding the packet [16]. Most of the earlier work has concentrated on properties based on the shortest path and hence we address the other extreme in this paper [14].

The determination of the shortest path between two nodes is prohibitively expensive for networks with a large number of connections or networks where nodes and connections are added or removed dynamically. Examples of such networks are social networks, telephone networks, the internet, etc. Another problem with the determination of the shortest path is the incomplete knowledge of the network. Hence, it is clear that an alternative method of generating a path (which need not be the shortest) becomes necessary in these networks. Our analysis of the random walk shows that the average access time (or the first passage time) between two nodes varies as $O(n)$, for small-world geometry, where n is the number of nodes. It is thus beneficial to consider a network with random routing or switching, particularly if it has small-world properties. Thus the random routing emerges as both a practical and a computationally cheaper method for large networks.

It is interesting to note that the normal practice of sending packets through the internet does have a random element in it. If a computer has k connections, depending on the address to which the packet is to be sent, some of the connections are chosen in a deterministic way, while if the packet is not addressed to one of these deterministic values it is sent via one of the remaining connections randomly. Such a choice is normally based on the path of least traffic and need not correspond to the shortest path. There is usually an upper limit to the number of such steps that the packet takes [19]; if the node is not reached within these steps the address is treated as untraceable. Our results now show that such a random routing can be effective for small-world geometries.

In Sec. II we discuss analytical and numerical results for the *average access time* of the random walk on a one-parameter family of graphs ranging from the regular case to the random case. We introduce an *independent step approximation* that allows us to get analytical expressions for the average access time. We discuss these results in Sec. III. It is found that the random walk results are similar to the shortest path results. Thus from the nature of the outcome of an experiment it may be difficult to conclude whether the spread was random or along the shortest path. An important consequence of this result can be in the routing and switching in very big networks. Random routing is a promising method, particularly if the network has small-world properties. Section IV summarizes the results.

II. RANDOM WALK ON SMALL-WORLD GRAPHS

The random walk on a graph is performed as follows: We start with a fixed node (say i) and at each step make a jump

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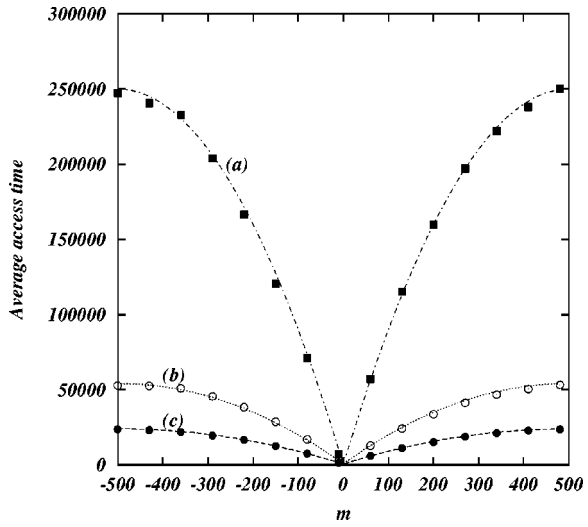

FIG. 1. The plot of average access time vs the distance m of a site from the starting point $(m=0)$ for regular graphs. Curves *a*, *b*, and *c* correspond to $k=1$, 3, and 5, respectively. The points are the result of simulations of random walks on a graph of size 1000. The lines are the analytical and scaling results obtained using Eqs. (4) and (5).

to a node connected to i with uniform probability $1/d(i)$, where $d(i)$ is the degree of the node i, thus performing a random walk. Such a random walk gives a finite *Markov chain* [17]. One of the most important quantities of interest in a finite Markov chain is the average access time. Let $D_{i,j}$ be the average access time, defined as the first passage time to the node j if the walk starts from the node i. We denote $D_j=D_{0,j}$.

We perform random walks on the family of graphs generated by the algorithm given by Watts and Strogatz [1]. The prescription gives a one-parameter family of graphs that interpolates between the regular case and the random case. We refer to this family as the family of small-world networks or graphs. The regular graph (denoted by $p=0$) is a graph with n vertices on a circle with each node connected to $2k$ nearest neighbors. The parameter k is suitably chosen to keep the graph sparse but connected. The other elements of the family are obtained by random rewiring of each edge in the graph with probability p. It is seen that the small-world behavior is prominent around the parameter value $p=0.01$, i.e., when only 1% of edges are rewired. $p=1$ corresponds to the random case [20].

A. The regular case $(p=0)$

Figure 1 shows the results for the average access time in a simulation of a random walk on the regular graph with 1000 nodes for several values of k. The average access time shows a linear behavior for small m and a quadratic nature for large m due to the circular topology. The lines are the analytic curves obtained as follows.

From the expectation values of conditional events, we can easily write the recursion relation for average access time for the walk starting from node i to node m, denoted by $D_{i,m}$, as [18]

$$D_{i,m}=\frac{1}{2k}\sum_{j=1}^{k}(D_{i+j,m}+D_{i-j,m})+1. \tag{1}$$

This is a $2k$th order difference equation for $D_{i,m}$. We first consider the case $k=1$. In this case the above recursion relation reduces to a quadratic form given by

$$D_{i,m}=\tfrac{1}{2}(D_{i+1,m}+D_{i-1,m})+1. \tag{2}$$

The equation can be solved using standard techniques [18] and the solution is given by

$$D_{i,m}=-(m-i)^2+A(m-i)+B, \tag{3}$$

where A and B are constants. The constants are determined using the boundary conditions $D_{m,m}=0$ and $D_{m-n,m}=0$. Hence,

$$D_{i,m}=-i^2+(2m-n)i-m^2+mn.$$

Without loss of generality, we assume that the walk starts from $i=0$. So the average access time for site m starting from zero is

$$D_m=-m^2+mn. \tag{4}$$

Curve *a* in Fig. 1 shows both the analytical and numerical results for the case $k=1$. The linear and quadratic parts are clearly seen. This behavior of D_m is the result of the normal one-dimensional random walk and the effect of the boundary condition. Note that $(\partial D_m/\partial m)_{m=n/2}=0$, as it should for the cyclic boundary condition.

For a general value of k, it is not possible to solve the difference equation (1) exactly. The numerical simulations show that the nature of the curves for different k is the same as for $k=1$. This suggests that there may be a scaling relation for general k. Using numerical data fitting we find that the following scaling relation fits the data reasonably well:

$$D_m^k(n)\approx[1+\mu\ln(k)]D_{m/k}^1\left(\frac{n}{k}\right), \tag{5}$$

where D_m^k is the average access time from site 0 to site m on the graph with $2k$ nearest neighbors and $\mu=0.86$ [21].

A possible explanation for such a scaling can be obtained by reducing the graph by obtaining the quotient using a subgraph of size k. Figure 2 explains this procedure for obtaining a reduced graph. The outer graph in Fig. 2 is of size 30 and the inner reduced graph is the graph that has only nearest neighbor connections. While the walk on the reduced graph is described by Eq. (4), the coefficient probably comes from the average time spent in each block. Note that the number of blocks that the walker has to pass is m/k.

From Fig. 1 we see that the scaling relation (5) shows an excellent matching with the numerical results for various values of k.

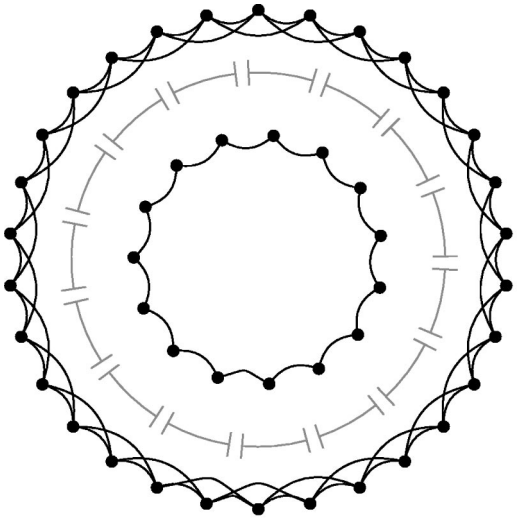

FIG. 2. The procedure for reducing the regular graph with n nodes and $2k$ neighbors to a graph with n/k nodes and two neighbors. The outer graph has 30 nodes and the inner reduced graph has 15 nodes.

B. The random case ($p=1$)

For the completely random case the access time becomes independent of m and $D_{i,j}=n-1,\forall i,j$. This result can be obtained as follows.

We note that to calculate the average access time one must consider an ensemble of graphs for a given p. The average access time is obtained by first averaging over several realizations of the random walk on a given graph and then over various members of the ensemble of graphs. We now introduce an independent step approximation where we assume that the order of these two averages is interchanged. Thus in this approximation each step of the random walk is averaged over all the realizations of the graphs. This approximation is a kind of mean field approximation done in statistical mechanics.

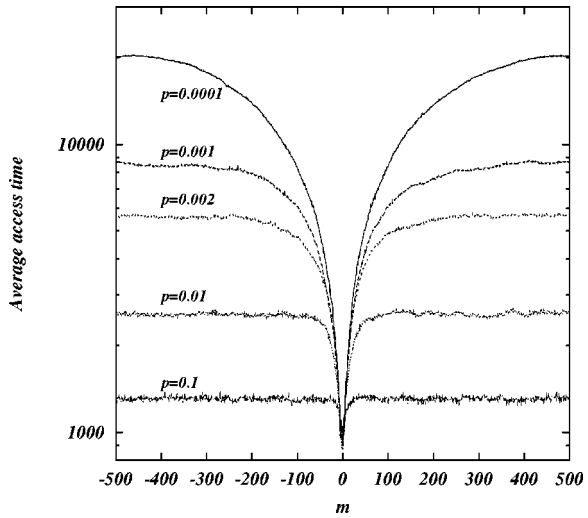

FIG. 3. The plot of average access time vs the distance m from the starting point $m=0$ for a graph with 1000 nodes and $k=5$, generated for various values of parameter p. The average access time clearly shows a crossover from regular to random behavior with increasing m. Note the logarithmic scale for the average access time.

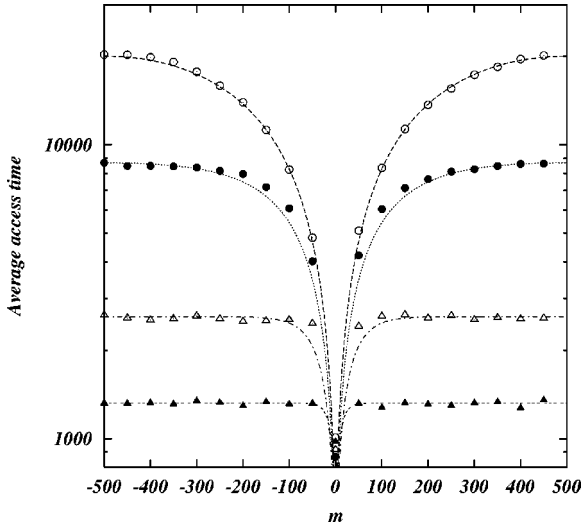

FIG. 4. Plot of average access time vs the distance m from the starting point of the random walk. The points are the numerical results for $n=1000$, $k=5$, and $p=0.0001$ (open circles), $p=0.001$ (solid circles), $p=0.01$ (open triangles), and $p=0.1$ (solid triangles). The lines are obtained using the scaling relation Eq. (12).

Let $P_{i,j}$ be the probability of reaching site j from site i in one step. It is obvious that $\Sigma_i P_{i,j}=1$. Using the independent step approximation, we get

$$P_{i,j}=\frac{1}{n-1}.$$

Thus the probability of reaching a site m at any time step is $1/(n-1)$. Hence, the probability of reaching site m for the first time in t time steps is

$$P_m(t)=\left(1-\frac{1}{n-1}\right)^{t-1}\frac{1}{n-1}. \tag{6}$$

Thus D_m is given by

$$D_m=\sum_t tP_m(t)$$

$$=n-1. \tag{7}$$

We get the interesting and simple result that the average access time for any two sites of a random graph is of the order of the total number of nodes or sites of the graph. We have verified this result numerically.

C. The intermediate case ($0<p<1$)

Figure 3 shows the results for average access time on a graph with 1000 nodes and $k=5$. Two distinct behaviors of average access time can be identified from Fig. 3. For small values of m, the behavior is similar to that of the regular case, while for larger values of m the average access time

saturates and behaves like that of a random graph (see Sec. II B). As p increases the saturation of the average access time becomes more prominent and the crossover from regular to saturation behavior takes place at smaller and smaller values of m. For $p=0.01$, which corresponds to small-world behavior [1], the average access time behavior is almost the same as that of the random graph, i.e., a constant of the order of n, except for small values of m of the order of k. The saturation of D_m for large values of m is clearly because of the random long range connections. Due to these connections we are able to reach the sites at the other end more quickly, thus leading to the saturation of D_m.

To obtain analytical estimates of the average access time we again make use of the independent step approximation defined in Sec. II B. In this approximation each step in the walk is taken with a probability $(1-p)$ to one of the nearest $2k$ sites and with probability p to the remaining sites. In analogy with the $p=0$ case we write a recursion relation for $D_{i,m}$ (see the Appendix),

$$D_{i,m}=(1-p)\left[\frac{1}{2k}\sum_{l=0}^{k}(D_{i+l,m}+D_{i-l,m})+1\right]+p\left[\frac{1}{n-2k-1}\sum_{\substack{l=0\\ l\neq i-k,\ldots,i+k}}^{n-1}D_{l,m}+1\right]. \tag{8}$$

This is a $2k$th order difference equation. As in the case of $p=0$, we first consider the case $k=1$. The recursion relation becomes

$$D_{i,m}=(1-p)\left[\tfrac{1}{2}(D_{i+1,m}+D_{i-1,m})+1\right]+p\left[\frac{1}{n-3}\sum_{\substack{l=0\\ i\neq i,i-1,i+1}}^{n-1}D_{l,m}+1\right], \tag{9}$$

which can be written as

$$D_{i,m}=(1-p)\left[\tfrac{1}{2}(D_{i+1,m}+D_{i-1,m})+1\right]+p\left[\frac{1}{n-3}\sum_{l=0}^{n-1}D_{l,m}+1-\frac{1}{n-3}(D_{i,m}+D_{i+1,m}+D_{i-1,m})\right].$$

With a little algebra, we finally get

$$D_{i,m}=\xi(D_{i+1,m}+D_{i-1,m})+\zeta, \tag{10}$$

where

$$\xi=\frac{(n-3)-p(n-1)}{2(n-3+p)}$$

and

$$\zeta=\frac{p\sum_{j=0}^{n-1}D_{i,m}+n-3}{n-3+p}.$$

Note that the sum $\Sigma_{j=0}^{n-1}D_{i,m}$ is independent of the site index and can be treated as a constant to be determined self-consistently from the solution. We solve the different equation (10) using the standard methods [18]:

$$D_{i,m}=A\theta_+^{(m-i)}+B\theta_-^{(m-i)}-\frac{\zeta}{2\xi-1},$$

where

$$\theta_\pm=\frac{1}{2\xi}(1\pm\sqrt{1-4\xi^2}).$$

The constants A and B are determined using the boundary conditions $D_{m,m}=0$ and $D_{m-n,m}=0$. The solution is given by

$$D_m=\frac{\zeta}{2\xi-1}\left[\frac{\theta_-^{-m}\theta_+^{-n}-\theta_+^{-m}\theta_-^{-n}-\theta_-^{-m}+\theta_+^{-m}}{\theta_+^{-n}-\theta_-^{-n}}-1\right], \tag{11}$$

where without loss of generality we have put the starting point as $i=0$. We note that D_m has an exponential dependence on m, which is different from the polynomial dependence for $p=0$. This exponential dependence is responsible for the saturation of D_m as observed in Fig. 3 for large m. This point is further discussed in the next section.

The sum $\Sigma_{m=0}^{n-1}D_m$ occurring in ζ is calculated by summing Eq. (11) for all values of m and is given by

$$\sum_{m=0}^{n-1}D_m=\frac{(n-3)}{p\left\{1+n/\sum_{m=1}^{n-1}\left[(\theta_-^{-m}\theta_+^{-n}-\theta_+^{-m}\theta_-^{-n}-\theta_-^{-m}+\theta_+^{-m})/(\theta_+^{-n}-\theta_-^{-n})-1\right]\right\}}.$$

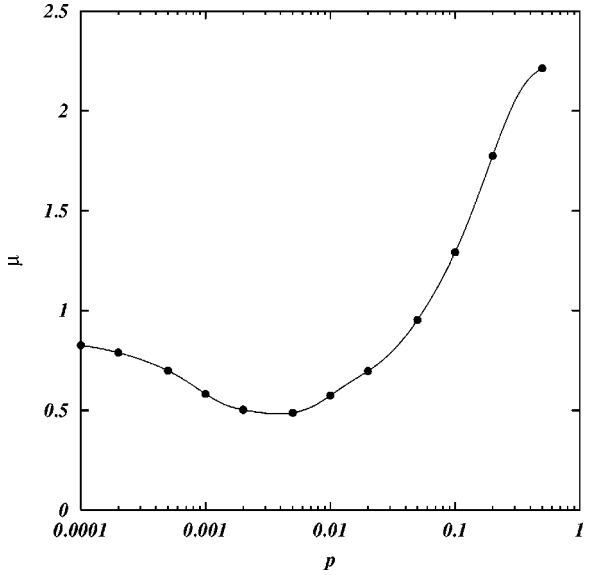

FIG. 5. Plot of μ as a function of p.

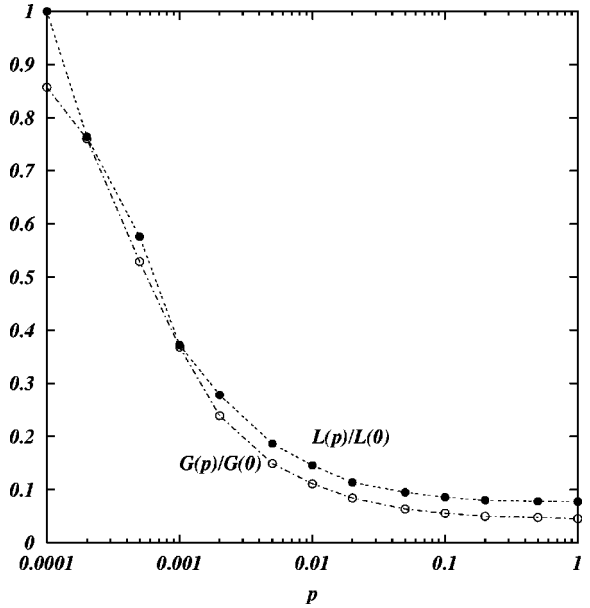

FIG. 6. Plot of $G(p)/G(0)$, i.e., the normalized average access time for the diametrically opposite point of the graph, vs p. The plot also shows characteristic length $L(p)/L(0)$ as a function of p. Both quantities show a similar behavior. In particular, the sharp drop near the small-world regime ($p \approx 0.01$) to the value corresponding to the random case is clearly seen in the figure.

Note that Eq. (8) and further analysis exhibits random behavior as $p \to (n-2k-1)/(n-1)$, rather than $p=1$ (see Appendix).

For general values of k, we again write a scaling relation similar to that for the case $p=0$:

$$D_m^k(n,p) \approx [1+\mu(p,k)\ln(k)] D_{m/k}^1\left(\frac{n}{k},p\right). \qquad (12)$$

Figure 4 shows the match of the numerical data with the analytic expressions. The parameter μ has a weak dependence on k for nonzero p, which we have neglected in further analysis. Figure 5 shows the behavior of μ for the various values of p. In Fig. 4 it is clearly seen that for small and large values of p the match between numerical results and Eqs. (11) and (5) is quite good, but for intermediate values of p there is a considerable deviation from the numerical results for small values of m. This fact can be understood as follows.

For small values of p, i.e., when $p \approx 10^{-4}$, the graph is nearly regular and the blocks of size k are nearly completely connected graphs, but as p increases the probability of leaving the block randomly for a faraway point increases, giving rise to higher average access times for the nearby points in the block than in the analytical values. Again, for high values of p the expression has a good match because as far as the average access time is concerned the completely connected block and a random block behave in a similar way.

III. DISCUSSION

It is interesting to compare the limiting behaviors of intermediate cases with those for the regular and random cases. Equation (9) reduces to Eq. (2) as $p \to 0$. However, the solution [Eq. (11)] does not smoothly reduce to Eq. (4). This is because of the degeneracy in the roots of the quadratic indicial equation Eq. (2) for $p=0$, which is lifted for nonzero p. Thus the solution changes from a polynomial to an exponential in site index as p becomes nonzero. This explains the reason behind the saturation of the solution for nonzero p for asymptotic m, which cannot be obtained for $p=0$. This change in the behavior is probably a reflection of the fact that there is a phase transition at $p=0$ [13].

For the random case we again find a similar situation. As $p \to (n-3)/(n-1)$, which corresponds to the random case (see the Appendix), Eq. (9) reduces to $D_m = \zeta$ and is consistent with the solution $D_0 = 0$ and $D_m = n-1$, $m \neq 0$. However, the solution [Eq. (11)] does not smoothly reduce to a constant function because one of the roots (θ_+) diverges.

It is interesting to consider the effect of the boundary on average access time. This effect is quite important for the $p=0$ case where the quadratic dependence on m is obtained due to the boundary condition (4). For nonzero p, the concept of a boundary starts breaking down due to long range connections. One is able to reach the other end more quickly due to these random connections. In the extreme case of random networks there is no boundary. The breaking of the boundary is reflected in the saturation of D_m for large m. In fact, for $p \simeq 0.01$ (or for larger p), which is approximately the small-world value, the boundary has almost vanished and the behavior is similar to that for a random network (see Fig. 3) except for very small m. Thus for the family of small-world graphs the boundary is not important for most of the range of p.

Next, we consider the average access time for the diametrically opposite node, i.e., $G(p) = D_{n/2}(p)$ (Ref. [9] discusses some results for diametrically opposite nodes). This quantity is of interest as it should be correlated with the average cover time for the graph. Figure 6 shows the behavior of the access time of the diametrically opposite point normalized to that of the $p=0$ case, i.e., $G(p)/G(0)$, as a

function of p. The figure also shows the graph of characteristic length $L(p)/L(0)$ [1], which is the average shortest path between any two sites. Both the curves in the graph, corresponding to the shortest path and the random walk, show similar behavior. This observation has interesting consequences. Random spread on a small-world network considerably reduces the access time compared to that of the regular graph, as in the case of the shortest path spread. This can be very useful in applications such as routing and switching where random routing is cheaper. The determination of the shortest path is generally very expensive for large or dynamically changing networks. Also in many cases complete information on the network is not available. Examples of such networks are social networks, telephone networks, the internet, etc., where the number of nodes is very large and also dynamically changing. The dynamics of the network includes additions and deletions of nodes and connections as well as temporary failure of some nodes and connections. In such cases random routing may be more effective and cheaper, particularly if the graph has the small-world geometry where the average access time varies as $O(n)$. We note that it may be necessary in a random routing to place an upper limit on the path length as is done in network connections (see the Introduction) since there could be some paths of very long length although the average is small.

In the calculations we have introduced an independent step approximation, where we average over the various realizations of graphs at each time step. This is a kind of mean field approximation and has allowed us to obtain recursion relations for the average access time. We expect this approximation to be reasonably good for the random case. Even for intermediate cases, the results obtained by this approximation are in good agreement with the numerical values.

For general values of k the recursion relations for average access time cannot be solved. However, the behavior of the average access time shows an interesting scaling relation in k. As discussed in Sec. II A the scaling relation corresponds to a reducing procedure where a graph of n nodes with $2k$ connections is scaled to a graph of m/k nodes with nearest neighbor connections and preserving the far edges [2]. The scaling gives a very good fit to the numerical data except for small values of m, when $p \geqslant 0.001$.

Lastly we consider the effect of the number of nodes on our results. We have studied lattices up to size $n=20\,000$. We have not noticed any significant deviation from the results reported here.

IV. SUMMARY

In this paper, we have studied a random walk on the family of small-world graphs. For the regular case the average access time shows linear behavior for small distances and a quadratic nature for large distances. An interesting scaling relation in k is observed for the average access time. For the random case the average access time is $(n-1)$ and is independent of distance and k. An independent step approximation has enabled us to get the analytical result for the average access time. The same approximation allows us to write a recursion relation for the intermediate values of p. For intermediate cases the average access time shows a crossover from regular to random behavior with increasing distances.

The normalized average access time of diametrically opposite nodes shows almost identical behavior to that of the characteristic length as a function of p. This observation might be very important in several applications where the number of nodes in a graph is very large or complete information about the graph is not available. In these cases random routing or switching may be beneficial.

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APPENDIX

Derivation of Eq. (9)

We assume that the probability of breaking an edge is p. We identify that the contribution to the average access time of site j comes from three types of events.

(1) All the $2k$ neighbors of site j are connected to j. The probability associated with this event is $(1-p)^{2k}$.

(2) Some of the neighbors are connected to j (say $2k-r$) and r are connected to the faraway sites. The probability associated with this event is $p^r(1-p)^{2k-r}$.

(3) None of the neighbors of j are connected to j. The probability associated with this event is p^{2k}.

By the independent step approximation the degree of each node is $2k$. This enables us to write a recursion relation for average access time using the properties of the expectation value of conditional probability [18]. We write

$$D_{i,m}=\sum_{r=0}^{2k} {}^{2k}C_r p^r (1-p)^{2k-r} \times \left\{ \frac{2k-r-1}{2k}\left[\sum_{l=1}^{k}(D_{i-l,m}+D_{i+l,m})\right] + \frac{r}{n-2k-1}\sum_{\substack{l=0 \\ l\neq i-k,\ldots,i+k}}^{n-1} D_{l,m}\right\}+1, \tag{A1}$$

$$=(1-p)\left[\frac{1}{2k}\sum_{l=0}^{k}(D_{i+l,m}+D_{i-l,m})+1\right] + p\left[\frac{1}{n-2k-1}\sum_{\substack{l=0 \\ l\neq i-k,\ldots,i+k}}^{n-1} D_{l,m}+1\right]. \tag{A2}$$

In the above calculation we have assumed that the edge broken with probability p is not rewired to one of the $2k$ nearest neighbors. Due to this the random graph limit corresponds to $p=(n-2k-1)/(n-1)$ instead of $p=1$. This can be seen by equating the probability of a connection to a nearest neighbor to the probability of a connection to any of the other sites,

$$\frac{(1-p)}{2k}=\frac{p}{n-2k-1}.$$

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[18] W. Feller, *An Introduction to Probability Theory and Its Applications*, 3rd ed. (Wiley, New York, 1968).
[19] The maximum number of steps a packet can take is determined by the size of the ''Time To Live'' field in the packet header, which is 1 byte long, i.e., the maximum time to live is 255 steps.
[20] In the rewiring algorithm of Watts and Strogatz the random graph defined by $p=1$ is not truly random. However, as far as spread is concerned the small-world network ($p=0.01$) itself is a good enough approximation to the random network. Hence, the $p=1$ case is adequate to describe the random behavior.
[21] We have also tried other types of scaling relations, in particular, a scaling relation of the form $D_m^k(n) \approx k^{\mu} D_{m/k}^1(n/k)$. However, we find that the relation (5) gives a better fit to the numerical data.